MATH 800: Commutative Algebra – Lecture 16 – Nov. 01, 2013

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1 Localization and Prime Ideals

Definition (notation)**.** Let R and S be as last time (recall that S denotes a multiplicatively closed subset of a commutative ring R). Let $\mathrm{Spec}_S(R)$ be the set of prime ideals of R which are disjoint from S.

Proposition. *The ideal correspondence from last time is a bijection of lattices when restricted to* $Spec(S^{-1}R) \to Spec_S(R)$. *This is given by* $B \longmapsto \{a \in R : a/1 \in B\}$, *with the inverse map given by sending* $A \in Spec_S(R)$ *to* $S^{-1}A$.

Proof. First check if $\mathfrak{p} \in \mathrm{Spec}_S(R)$ implies $S^{-1}\mathfrak{p} \in \mathrm{Spec}(S^{-1}R)$. Suppose $r_1/s_1, r_2/s_2 \in S^{-1}\mathfrak{p}$ then there is $s \in S$ such that $r_1 r_2 \in \mathfrak{p}$. But $\mathfrak{p} \cap S = \varnothing$ so $r_1 r_2 \in \mathfrak{p}$ by primality. So again by primality, $r_1 \in \mathfrak{p}$ or $r_2 \in \mathfrak{p}$ so $r_1/s_1 \in S^{-1}\mathfrak{p}$ or $r_2/s_2 \in S^{-1}\mathfrak{p}$.
Next check if $B \in \mathrm{Spec}(S^{-1}R)$ then $A = \{a \in R : a/1 \in B\} \in \mathrm{Spec}_S(R)$. Suppose $ab \in A$. Then $ab/1 \in B$ so $(a/1)(b/1) \in B$. So by primality $a/1 \in B$ or $b/1 \in B$. So $a \in A$ or $b \in B$. Lastly we must check that composition both ways is the identity. We saw last time that if B is just any ideal of $S^{-1}R$, then $S^{-1}\{a : a/1 \in B\} = B$. So it remains to show for $\mathfrak{p} \in Spec_S R$ the ideal $A = \{a : a/1 \in S^{-1}\mathfrak{p}\}$ is equal to $\mathfrak{p}$. Take $a \in \mathfrak{p}$, then $a/1 \in S^{-1}\mathfrak{p}$ so $a \in A$. Take $a \in A$ so that $a/1 \in S^{-1}\mathfrak{p}$ so there exists $s \in S$ such that $as \in \mathfrak{p}$. But $\mathfrak{p} \cap S = \varnothing$ and $\mathfrak{p}$ is prime so $a \in \mathfrak{p}$. ☐

Corollary. *We always have Kdim* $S^{-1}R \leq$ *Kdim* R.

Proof. Given $\mathfrak{p} \in \mathrm{Spec}_S(R)$ any chain with $\mathfrak{p}$ at the top consists only of prime ideals disjoint from S so height of $\mathfrak{p}$ is the same as height of $S^{-1}\mathfrak{p}$. So the result follows. ☐

Proposition. *R and S as before with $S \subset C$. If R is integral over C, then $S^{-1}R$ is integral over $S^{-1}C$.*

Proof. Take $r/s \in S^{-1}R$. Then $r/1$ is integral over $S^{-1}C$ by the same polynomial which makes r integral over C. Also, $1/s \in S^{-1}C$ so it is certainly integral. Hence, $r/s = (r/1)(1/s)$ is integral over $S^{-1}C$. This gives the desired result. ☐

2 Local Rings

Definition (notation)**.** Take $\mathfrak{p} \in \operatorname{Spec}(R)$ and localize at $S = R \setminus \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the localization of R at $\mathfrak{p}$, also denoted $R_\mathfrak{p}$. Likewise $\mathfrak{p}_\mathfrak{p}$ is the image of $\mathfrak{p}$ in $R_\mathfrak{p}$.

Proposition. *Let $\mathfrak{p}, R$, and S be a above. Then*

1. *$R_\mathfrak{p}$ has a unique maximal ideal $P_\mathfrak{p}$.*
2. *$\mathfrak{p}$ and $R_\mathfrak{p}$ have the same height, which is equal to Kdim $R_\mathfrak{p}$.*

Proof. For the first assertion, let M be a maximal ideal of $R_\mathfrak{p}$. Let $A = \{a : a/1 \in M\}$ and note $A \cap S = \varnothing$. But then $A \subseteq \mathfrak{p}$. So $M \subseteq \mathfrak{p}_\mathfrak{p}$. But M is maximal so $M = \mathfrak{p}_\mathfrak{p}$. For the second assertion, note the first equality follows since the prime ideals contained in $\mathfrak{p}$ are preserved in $R_\mathfrak{p}$. The second equality now follows by the first assertion. □

Example: Let $R = \mathbb{Z}$. Let $\mathfrak{p} = p\mathbb{Z}$ for some prime number p. Then $S = \mathbb{Z} \setminus \mathfrak{p} = \{n \in \mathbb{Z} : \gcd(n,p) = 1\}$. So $\mathbb{Z}_\mathfrak{p} = \{m/n : m \in \mathbb{Z}, \gcd(n,p) = 1\}$.

Definition (Local Ring)**.** A commutative ring R is said to be a local ring if R has a unique maximal ideal.

Observe that the localization of a commutative ring at a prime ideal $\mathfrak{p}$ is clearly a local ring.

Proposition. *The following are equivalent.*

1. *R is a local ring.*
2. *The set of all non-invertible elements of R is an ideal.*
3. *The sum of any two non-invertible elements is non-invertible.*
4. *If $a + b = 1 \in R$, then a or b is invertible in R.*

Proof. Note statement 2 clearly implies 3. Let us begin by showing that 3 implies 4. Note the contrapositive of 3 is: if $a + b$ is invertible, then a or b is invertible. So 4 is a special case of the contrapositive of 3. Now to show 4 implies 3, suppose $a + b = u$ for some unit u in R. Then $au^{-1} + bu^{-1} = 1$. So by statement 4 either au^{-1} or bu^{-1} is invertible, so a or b is invertible. This implies the contrapositive of 3, and hence 3 itself. Now to see that 3 implies 2, take a a non-invertible element of R and let $r \in R$. Consider ra. If ra had an inverse, then $ra(ra)^{-1} = 1$ so that $a(r(ra)^{-1}) = 1$, a contradiction. This shows that ra is non-invertible. Hence, the set of non-invertible element forms an ideal. Lastly, it remains to show that the first two statements are equivalent. To see that 2 implies 1, let P be the set of non-invertible elements. By assumption, P is an ideal and is maximal as any other element adjoined to it would give 1. If Q were another maximal ideal with $Q \neq P$, then Q contains an element not in P, and hence a unit, a contradiction. Lastly, to show that 1 implies 2, let P be the unique maximal ideal. Take $a \in R$ not invertible. Then Ra is an ideal so $Ra \subseteq P$, and so $a \in P$. □

Note: *1. If R is a local ring with maximal ideal P, then if $a \in P$ we have $1-a$ is invertible by statement 4 of the Proposition above.*

2. If R is affine, then it is Noetherian, and so R_P is Noetherian. But R_P is not affine.

Proposition. *Let R be a domain. Then $R = \bigcap_{P \text{ maximal ideal of } R} R_P$.*

Proof. Take $a \in \bigcap R_P$. Let $B = \{b \in R : ba \in R\}$. Note B is an ideal. Suppose $B \subsetneq R$. Then $B \subseteq P$ and P a maximal ideal of R. But $a \in \bigcap R_P$ so $a \in R_P$. This means $a = r/q$, where $r \in R$ and $q \notin P$. Then $qa \in R$ so $q \in B \subseteq P$, a contradiction. This gives $B = R$; $1 \in B$ and so $a \in R$. The other direction is trivial, and so we are done. □

Proposition (Nakayama's Lemma)**.** *Let R be a local ring with maximal ideal P. Let M be a non-zero finitely generated R-module. Then $PM \neq M$.*

Proof. Write $M = Ra_1 + \cdots + Ra_n$, for some $a_1, \ldots, a_n \in M$ with n minimal. Suppose to the contrary that $PM = M$. Then we can write $a_n = \sum_{j=1}^{n-1} p_j a_j$ for suitable choices of $p_1, \ldots, p_{n-1} \in P$. Then

$$(1-p_n)a_n = \sum_{j=1}^{n-1} p_j a_j,$$

for some $p_n \in P$. But $1-p_n$ is invertible so that a_n can be written in terms of remaining $n-1$ generators, contradicting the minimality of n. □

Corollary. *Let R, P, and M be as in Nakayama's Lemma. Then for every submodule $N \neq M$, we have $N + PM \neq M$.*

Proof. Apply Nakayama's Lemma to M/N to get $P(M/N) \neq M/N$. So $N + PM \neq M$. □

Corollary. *Let R, P, and M be as above. Let $B \subseteq M$ be such that the image of B in M/PM spans M/PM (as a vector space over R/P). Then B spans M.*

Proof. Let $N = \sum Rb_j$. The image of N in M/PM is M/PM. So $N + PM = M$. Applying the previous corollary gives $N = M$. □

Corollary. *Let R be a domain and let $P \in Spec(R)$. Suppose A is a non-zero ideal of R with $A \subseteq P$ such that A is finitely generated as an R-module. Then $PA \subsetneq A$.*

Proof. If $PA = A$, then $P_P A_P = A_P$ in R_P, contradicting Nakayama's Lemma. □

3 Artinian Implies Noetherian for Commutative Rings

Recall that we noted last time that if a module is both Artinian and Noetherian, then it must have a composition series. We now prove that the converse holds when the underlying ring is commutative.

Proposition. *Let M be an R-module. If M has a composition series, then M is both Artinian and Noetherian.*

Proof. Since M has a composition series, say of length n, any other composition series can be refined to a composition series that is equivalent and so has length at most n. So M is both Artinian and Noetherian. □

Theorem 3.1. *If R is an Artinian commutative ring, then R is Noetherian.*

Proof. Suppose R is Artinian. Consider all ideals of R which are products of maximal ideals of R. Since R is Artinian, we may choose a minimal such ideal, say J. We would like to show that $J = 0$. First, note if M is any maximal ideal of R, then $MJ = J$ by minimality of J. Consequently, $J \subseteq M$. Otherwise, there exists $j \in J$ with $j \notin M$. This means $j \notin MJ \subseteq M$, a contradiction. Second, J^2 is also a product of maximal ideals so again $J^2 = J$, by minimality of J. Now suppose $J \neq 0$. Consider the set of all ideals not annihilated by J; choose I minimal with respect to this property. Then

$$0 \neq IJ = IJ^2 = (IJ)J,$$

so $IJ = I$, by minimality of I. In particular, there exists $f \in I$ with $fJ \neq 0$. So the minimality of I implies $I = (f)$; i.e., I is generated by f. Hence, there exists $g \in J$ with $fg = f$ (recall we had $IJ = I$). So $(1 - g)f = 0$. But J is contained in every maximal ideal, and so g is also contained in every maximal ideal. But then $1 - g$ is contained in no maximal ideal; in other words, $1 - g$ is a unit. This immediately gives $f = 0$, contradicting the assumption that $J \neq 0$.
Now we have $M_1 \cdots M_t = 0$ for some maximal ideals M_i of R. Consider, for each $s \geq 0$,

$$(M_1 \cdots M_s)/(M_1 \cdots M_{s+1}).$$

Note this is a vector space over R/M_{s+1}. Since any subspace is a submodule, this corresponds to an ideal of R containing $M_1 \cdots M_{s+1}$. But R is Artinian so the vector space is Artinian, and thus is finite dimensional over R/M_{s+1}. But this means it has a composition series. Building these together we obtain a composition series for R; i.e., R is Noetherian. □